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Gauge theory based upon solvable Lie algebras

Susumu Okubo
Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, USA

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Abstract. It has been demonstrated that mathematically consistent Yang–Mills gauge theories can be constructed on the basis of a class of general Lie algebras called quasiclassical, which contains reductive as well as a class of solvable Lie algebras. However, if we require that these theories should be ghost-free, then only the standard gauge theories based upon compact Lie algebras are allowed. Nevertheless, these solvable gauge theories may be relevant for some integrable models based upon the zero-curvature condition.

1. Introduction

It is often stated in the literature that the non-Abelian Yang–Mills gauge theory can be constructed only for semisimple Lie algebras. The standard argument is based upon the following observation. Let $t_a (a = 1, 2, \ldots, N)$ be a basis of a Lie algebra L with the multiplication table of

$$[t_a, t_b] = f^c_{ab} t_c. \tag{1.1}$$

Let $A^a_\mu(x)$ be the gauge field and set [1] as usual

$$A_\mu(x) = t_a A^a_\mu(x) \tag{1.2a}$$
$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)]. \tag{1.2b}$$

The equation of motion should then be given by

$$\partial^\lambda F_{\lambda\mu}(x) + [A^\lambda(x), F_{\lambda\mu}(x)] = 0. \tag{1.3}$$

We ordinarily assume the Lagrangian $L(x)$ to be

$$L(x) = \tfrac{1}{4}\,\mathrm{Tr}(\mathrm{ad}F_{\mu\nu}(x)\mathrm{ad}F^{\mu\nu}(x)) \tag{1.4}$$

where ad is the adjoint representation. The action principle on the basis of $L(x)$, leads to equation (1.3), if and only if the Killing form

$$g_{ab} = \mathrm{Tr}(\mathrm{ad}t_a \mathrm{ad}t_b) \tag{1.5}$$

is non-degenerate, i.e. it possesses its inverse g^{ab} satisfying

$$g_{ab} g^{bc} = \delta^c_a. \tag{1.6}$$

Because of Cartan's criteria [2] of semisimplicity, the non-degeneracy of g_{ab} is equivalent to the semisimplicity of L.

However, this reasoning is unsatisfactory as well as inaccurate for the following reasons. First, we cannot then formulate the Abelian gauge theory on the same footing, since the Killing form of any Abelian Lie algebra is identically zero. Ordinarily, we treat Abelian gauge theory in a slightly different fashion. Since the standard model based upon the $SU_C(3) \otimes SU_L(2) \otimes U(1)$ group is a gauge theory containing both semisimple and Abelian Lie algebras (which is known as reductive Lie algebras), it will be more desirable to treat both semisimple and Abelian gauge theories on an equal footing. Second, there may exist another Lagrangian which is quadratic in $F^a_{\mu\nu}(x)$ and which will reproduce equation (1.3) for some non-semisimple Lie algebras.

The purpose of this paper is to address these questions. We shall first show that we can indeed treat both Abelian and semisimple gauge theories in the same way. Second, we can construct gauge theories for a large class of purely solvable Lie algebras. However, the theory always has the defect of containing ghosts. Because of this, we conclude that only physically viable gauge theory is indeed the one based upon compact Lie algebras.

Nevertheless, the gauge theories based upon solvable Lie algebras may be of some use for other problems, such as integrable model [3] which utilizes the zero-curvature condition $F_{\mu\nu}(x) = 0$. We will prove these assertions in section 2.

2. Quasi-classical Lie algebras

We assume in this paper that L is a finite-dimensional Lie algebra over the complex field, unless it is stated otherwise. Let $\rho(t)$ for $t \epsilon L$ be a non-trivial representation matrix of L and set

$$g_{ab} = \mathrm{Tr}[\rho(t_a)\rho(t_b)]. \tag{2.1}$$

If L is simple, it is well known [2] that g_{ab} given by equation (2.1) is always proportional to g_{ab} defined by equation (1.5) with non-zero multiplicative constant. However, this is not correct in general. We will first quote the following theorem of Bourbaki [4].

Theorem 1. Any one of the following statements is equivalent to all the others.

(1) L is reductive, i.e. its adjoint representation is completely reducible.

(2) The derived Lie sub-algebra $L_1 = [L, L]$ is semisimple.

(3) L is a direct sum of a semisimple Lie algebra and an Abelian Lie algebra.

(4) L has a finite-dimensional representation ρ such that g_{ab} given by equation (2.1) is non-degenerate, i.e. it has its inverse g^{ab}.

(5) L has a finite-dimensional faithful representation which is completely reducible.

(6) The radical of L is the centre of L.

Especially, if L is a direct sum of a semisimple and an Abelian Lie algebra, there then exists a representation ρ such that g_{ab} constructed by equation (2.1) is non-degenerate (we shall return to this point shortly). Therefore, if we define a Lagrangian $L(x)$ now by

$$L(x) = \tfrac{1}{4}\,\mathrm{Tr}[\rho(F_{\mu\nu})\rho(F^{\mu\nu})] \tag{2.2}$$

instead of that given by equation (1.4), then it correctly reproduces the desired equation of motion equation (1.3).

Theorem 1 also implies that this method does not work for Lie algebras other than reductive ones. However, we can construct a more general gauge theory quadratic in $F^a_{\mu\nu}(x)$ in the following way. To this end, we need some preparations.

By a symmetric bilinear form $(.\,,.)$ in L, it implies the validity of

$$(x, y) = (y, x) \qquad x, y \epsilon L. \tag{2.3}$$

We say that it is an associative form if

$$([x, y], z) = (x, [y, z]) \tag{2.4}$$

holds valid for any $x, y, z \epsilon L$. Finally, the bilinear form is non-degenerate, if $(x, y) = 0$ for all $x \epsilon L$ implies $y = 0$ in L. In [5], a Lie algebra L which possesses a symmetric, bilinear, associative, non-degenerate, form (x, y) is called a quasiclassical Lie algebra. Because of theorem 1, any reductive Lie algebra is automatically quasiclassical, if we identify

$$(t_a, t_b) = \mathrm{Tr}(\rho(t_a)\rho(t_b))(= g_{ab}).$$

However, the converse is not true, as we can see from many examples given in [5–7]. Such Lie algebras have been utilized to construct a class of simple flexible Lie-admissible algebras [5–7]. They have also been used to obtain some solutions of the Yang–Baxter equation in [8].

Let L be a quasiclassical Lie algebra, spanned by N basis vectors, $t_1, t_2, \ldots, t_N$ satisfying equation (1.1). If we define g_{ab} by

$$g_{ab} = (t_a, t_b) \tag{2.5}$$

and identify $x = t_a$, $y = t_b$, and $z = t_c$ in equation (2.4), it gives

$$f_{abc} = f^{d}_{ab} g_{dc} \tag{2.6}$$

to be completely antisymmetric in three indices a, b, c. Conversely, if g_{ab} with its inverse g^{ab} satisfies such a property, it defines a quasiclassical Lie algebra by introducing $(.\,,.)$ in L by equation (2.5). This fact implies that the most general gauge theory must be based upon a quasiclassical Lie algebra. Hence, we assume hereafter that L is quasiclassical.

Let G be the Lie group obtained by exponentiating L, and set

$$g = \exp t \epsilon G \tag{2.7}$$

for a element $t \epsilon L$. Then by equation (2.4), it is easy to see

$$(g^{-1}xg, y) = (x, gyg^{-1})$$

for any $x, y \epsilon L$. Replacing y by $g^{-1}yg$, we find

$$(g^{-1}xg, g^{-1}yg) = (x, y). \tag{2.8}$$

Changing notations, we reserve, for a while, the symbol x for the spacetime coordinate, and set

$$L(x) = (F_{\mu\nu}(x), F^{\mu\nu}(x)) = g^{ab} F_{\mu\nu,a}(x) F^{\mu\nu}_{,b}(x). \tag{2.9}$$

Let $g(x)$ given by

$$g(x) = \exp[\omega^a(x) t_a] \tag{2.10}$$

be the coordinate-dependent element of G, where $\omega^a(x)$ are functions of the spacetime coordinate x^μ. In view of equation (2.8), the Lagrangian $L(x)$ of equation (2.9) is invariant under the local gauge transformation

$$A_\mu(x) \to A'_\mu(x) = g^{-1}(x) A_\mu(x) g(x) - g^{-1}(x) \partial_\mu g(x) \tag{2.11a}$$

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = g^{-1}(x) F_{\mu\nu}(x) g(x). \tag{2.11b}$$

Moreover, since g_{ab} is non-degenerate, the Lagrange equation of motion based upon equation (2.9) will correctly reproduce equation (1.3). In conclusion, we can construct a mathematically consistent Yang–Mills gauge theory, if and only if the underlying Lie algebra is quasiclassical.

Before going into further details, we will state the following theorem [5] which characterizes the quasiclassical Lie algebra.

Theorem 2. A necessary and sufficient condition that a Lie algebra L is quasiclassical is that L possesses a second-order Casimir invariant

$$I_2 = g^{ab} t_a t_b \tag{2.12}$$

such that the symmetric matrix $g^{ab} (= g^{ba})$ has its inverse g_{ab} satisfying equation (1.6).

Proof. For I_2 given by equation (2.12), we calculate

$$[I_2, I_c] = (g^{bd} f^a_{dc} + g^{ad} f^b_{dc}) t_a t_b \tag{2.13}$$

so that $[I_2, I_c] = 0$, is equivalent to have

$$g^{bd} f^a_{dc} + g^{ad} f^b_{dc} = 0. \tag{2.14}$$

Multiplying $g_{aj} g_{bk}$, and changing indices suitably, equation (2.14) is shown to be also equivalent to

$$g_{ad} f^d_{bc} = -g_{bd} f^d_{ac}. \tag{2.15}$$

However, equation (2.15) is the same statement as to say that f_{abc} defined by equation (2.6) is totally antisymmetric in a, b, and c. Introducing $(.\,,.)$ by equation (2.5), it proves then that L is quasiclassical. Conversely if L is quasiclassical, we can prove $[I_2, I_c] = 0$ by reversing the argument. □

Theorem 2 gives a practical way of constructing g^{ab} and hence g_{ab}. For example, let L be a reductive Lie algebra. By theorem 1, L must be a direct sum of a semisimple algebra L_0 and an Abelian algebra L_1, i.e. $L = L_0 \oplus L_1$. Let us label the basis of L_0 and L_1 by $t_j (j = 1, 2, \ldots, n)$ for L_0 and by $t_\mu (\mu = 1, 2, \ldots, m)$ for L_1. Then, a second-order Casimir invariant I_2 satisfying the condition of theorem 2 is readily found to be

$$I_2 = \sum_{j,k=1}^{n} g^{jk} t_j t_k + \sum_{\mu=1}^{m} \xi^\mu (t_\mu)^2 \tag{2.16}$$

where $\sum_{j,k=1}^{n} g^{jk} t_j t_k$ is the second-order Casimir invariant of the semisimple Lie algebra L_0 and $\xi^\mu (\mu = 1, 2, \ldots, m)$ are arbitrary non-zero constants. This construction also shows that a reductive Lie algebra is quasiclassical by theorem 2.

In this connection, we may note the following fact. It is known [9] that an absence of the third-order Casimir invariant is intimately related to the absence of the triangle anomaly in gauge theories. If L is reductive with non-trivial Abelian part, it has always a third-order Casimar invariant of form

$$I_3 = I_2 I_1 + \sum_{\mu,\nu,\lambda=1}^{m} g^{\mu\nu\lambda} t_\mu t_\nu t_\lambda \tag{2.17}$$

for arbitrary constants $g^{\mu\nu\lambda}$ where I_2 is given by equation (2.16) with $\xi^\mu = 0$ and I_1 is the first-order Casimir invariant of the Abelian algebra L_1 given by

$$I_1 = \sum_{\mu=1}^{m} \eta^\mu t_\mu. \tag{2.18}$$

Here, η^μ are arbitrary constants such that at least one of them is non-zero. The condition that eigenvalues of any such I_3 should vanish for a given representation of L then leads to the familiar anomaly cancellation condition, although we will not go into detail here.

Returning to the original problem, Patera *et al* [10] have listed all algebraically independent Casimir invariants of all indecomposable real Lie algebras of dimensions three, four and five, as well as of all nilpotent Lie algebras of dimension six. From their list, we can easily see that Lie algebras $A_{4,8}$, $A_{4,10}$, $A_{5,3}$ and $A_{6,3}$ in their notation are quasiclassical solvable indecomposable Lie algebras, since all of them possess second-order Casimir invariants satisfying the condition of theorem 2. Moreover, $A_{5,3}$ and $A_{6,3}$ are nilpotent. From these, we can construct an infinite class of indecomposable solvable quasiclassical Lie algebras as we have shown in [5]. Here as an example, let us consider the Lie algebra $A_{4,8}$ whose non-zero commutation relations are specified by [10]

$$[t_2, t_3] = t_1, [t_2, t_4] = t_2, [t_3, t_4] = -t_3. \tag{2.19}$$

This algebra possesses the first- and second-order Casimir invariant of the form

$$\begin{aligned} I_1 &= t_1 \\ I_2 &= t_2 t_3 + t_3 t_2 - 2 t_1 t_4. \end{aligned} \tag{2.20}$$

Therefore, a non-trivial second-order Casimir invariant satisfying the condition of theorem 2 is given by

$$I_2' = I_2 + \lambda (I_1)^2 = g^{ab} t_a t_b \tag{2.21}$$

for an arbitrary constant λ. We then find

$$g^{ab} = \begin{pmatrix} \lambda & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad g_{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -\lambda \end{pmatrix}. \tag{2.22}$$

It is amusing to note that this algebra also appears in a study of non-decomposable representation of a Lie-super algebra in superspace [11]. The Lagrangian $L(x)$ given by equation (2.9) is calculated to be

$$\begin{aligned} L(x) &= \tfrac{1}{4} \sum_{a,b=1}^{4} g^{ab} F_{\mu\nu,a}(x) F^{\mu\nu}_{,b}(x) \\ &= \tfrac{1}{2} \{ F_{\mu\nu,2}(x) F^{\mu\nu}_{3}(x) - F_{\mu\nu,1}(x) F^{\mu\nu}_{4}(x) \} + \frac{\lambda}{4} F_{\mu\nu,1}(x) F^{\mu\nu}_{1}(x). \end{aligned} \tag{2.23}$$

Evidently, equation (2.23) does not yield a positive definite Hamiltonian for any gauge condition, so that the theory will contain ghost states when quantized. So we conclude that the theory is unphysical.

This appearance of ghost states is inevitable for any quasiclassical Lie algebra except for the case of compact Lie algebras which are automatically reductive [4]. We will prove this in the following theorem. We revert to the old notation now so that the symbol x refers to an element of L hereafter.

Theorem 3. Let L be a quasiclassical real or complex Lie algebra. Suppose that any nilpotent sub-Lie algebra B of L satisfying $(B, B) = 0$ implies $B = 0$ identically. Then, $L_1 = [L, L]$ is semisimple.

Proof. We will show that $L_1 = [L, L]$ is semisimple. Suppose that this is not true and L_1 has an Abelian ideal A. Then, we must have

$$[A, A] = 0 \qquad [A, L_1] \subseteq A.$$

Moreover, $[A, L_1]$ is clearly an Abelian sub-Lie algebra of L. We then calculate

$$([A, L_1], [A, L_1]) = ([[A, L_1], A], L_1)$$

but $[A, L_1] \subseteq A$ and hence, $[[A, L_1], A] = 0$. Thus, $([A, L_1], [A, L_1]) = 0$ so that $[A, L_1] = 0$ identically. We now note $[A, L] \subseteq [L, L] = L_1$ so that $[[A, L], A] = 0$. We then calculate $([A, L], [A, L]) = ([[A, L], A], L) = 0$ which leads to $[A, L] = 0$, since $[A, L]$ is also a nilpotent sub-Lie algebra of L because of $[[A, L], A] = 0$, and hence $[[A, L], [A, L]] = [A, [L, [A, L]]] \subseteq [A, L_1] \subseteq A$. Therefore, $(A, L_1) = (A, [L, L]) = ([A, L], L) = 0$. However, $A \subseteq L_1 = [L, L]$ so that this also requires $(A, A) = 0$, leading to $A = 0$ identically. This proves that $L_1 = [L, L]$ has no Abelian ideal, and hence is semisimple. □

Now we are in a position to prove that L must be a compact Lie algebra if the theory does not allow any ghost state. To this end, we must restrict ourselves to consideration of real (not complex) Lie algebras. The ghost-free condition requires that the matrix g_{ab} must be positive (or negative) definite, i.e.

$$g_{ab}\xi^a\xi^b = g^{ab}\xi_a\xi_b$$

is always positive for any real number ξ_a, all of which are not identically zero. Then, setting $x = \xi^a t_a$, this implies (x, x) to be positive for non-zero x. It evidently satisfies the condition of the theorem since $(x, x) = 0$ leads to $x = 0$. Therefore, theorems 3 and 1 imply L to be reductive, if we extend the field from real to complex fields. Since we are dealing with real algebra, the positiveness of $g_{ab}\xi^a\xi^b$ also implies L to be a compact Lie algebra [4].

In conclusion, the physically viable gauge theory must be solely based upon a compact Lie algebra. In this connection, we also note that Hickman *et al* [12] have studied Hamiltonian formulation of a gauge theory based upon a non-quasiclassical solvable Lie algebra by embedding it to larger semisimple Lie algebras in the unsuccessful hope that the quantization of such a theory may be possible in this way.

To end this paper, we remark that for some problems in mathematical physics the appearance or absence of ghost states is not relevant. For such cases, the general quasiclassical Lie algebra may be of some use. We have already noted that it has been used for studies of flexible Lie-admissible algebras as well as of some solutions of the Yang–Baxter equations. It may also be useful for studies of classical integrable models, where the zero curvature condition $F_{\mu\nu} = 0$ may be used.

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References

[1] Faddeev L D and Slavnov A A 1980 *Gauge Fields, Introduction to Quantum Theory* (Reading, MA: Benjamin)
[2] Humphreys J E 1972 *Introduction to Lie Algebras and Representation Theory* (New York: Springer)
[3] Das A 1989 *Integrable Models* (Singapore: World Scientific)
[4] Bourbaki N 1975 *Lie Groups and Lie Algebras, Part I* (Reading, MA: Addison-Wesley)
[5] Okubo S 1979 *Hadronic J.* **3** 1
[6] Myung H C 1986 *Malcev-Admissible Algebras* (Boston, MA: Birkhäuser)
[7] Benkart G M, Osborn J M and Britten D J 1981 *Hadronic J.* **4** 274
[8] Okubo S and Kamiya N 1996 Quasi-classical Lie-super algebras and Lie-super triple systems *University of Rochester Report* UR-1462 unpublished
[9] Okubo S 1977 *Phys. Rev.* D **16** 3528

[10] Patera J, Sharp R I, Winternitz P and Zassenhaus H 1976 *J. Math. Phys.* **17** 986
[11] Okubo S 1984 *Phys. Rev.* D **29** 269
Okubo S 1984 *Phys. Rev.* D **29** 1865 (erratum)
[12] Hickman M, Kozameh C, Lucey C and Newman E T 1987 *Phys. Rev.* D **36** 1740